NAHM'S CONJECTURE: ASYMPTOTIC COMPUTATIONS AND COUNTEREXAMPLES

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1. Introduction

Let $r \geq 1$ be a positive integer, A a real positive definite symmetric $r \times r$-matrix, B a vector of length r, and C a scalar. The series

$$F_{A,B,C}(q) \;=\; \sum_{n \in (\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}n^T A n + n^T B + C}}{(q)_{n_1} \ldots (q)_{n_r}}. \tag{1.1}$$

converges for $|q| < 1$. Here we use the notation $(a;q)_n := \prod_{k=1}^{n}(1 - aq^{k-1})$ for $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and the convention that the second argument is removed if it equals q (so $(q)_n = (q;q)_n = \prod_{k=1}^{n}(1 - q^k)$). We are concerned with the following problem due to Werner Nahm [2, 3, 4]: describe all such A, B and C with rational entries for which (1.1) is a modular form. In [5, 7, 8] it was solved by Michael Terhoeven and Don Zagier for $r = 1$ and the list contains seven triples $(A, B, C) \in \mathbb{Q}_+ \times \mathbb{Q} \times \mathbb{Q}$. We develop this approach for $r > 1$ and find several new examples of modular functions (1.1) already for $r = 2$.

Nahm has also given a conjectural criterion for a matrix A to be such that there exist some B and C with modular $F_{A,B,C}$ (see [4]). The condition for the matrix A is given in terms of solutions of a system of algebraic equations

$$1 - Q_i \;=\; \prod_{j=1}^{r} Q_j^{A_{ij}}\,, \qquad i \;=\; 1, \ldots, r\,. \tag{1.2}$$

In the last section we give several examples where the matrix A doesn't satisfy the condition but corresponding modular forms exist. Certainly, it doesn't mean that the conjecture is completely wrong, rather that its correct formulation is an interesting open question.

2. Asymptotical computations

Let us explain a method to compute the asymptotics of (1.1) when $q \to 1$. The idea comes from [8], where it is written in a very sketchy form. We denote the general term of the sum (1.1) by $a_n(q)$. Suppose $q \to 1$ and $n_i \to \infty$ so that $q^{n_i} \to Q_i$ for some numbers $Q_i \notin \{0, 1\}$. Then we have

$$\frac{a_{n+e_i}}{a_n} \;=\; \frac{q^{n^T A e_i + \frac{1}{2} e_i^T A e_i + e_i^T B}}{1 - q^{n_i+1}} \to \frac{Q_1^{A_{i1}} \ldots Q_r^{A_{ir}}}{1 - Q_i}\,,$$

where e_i is a vector whose all but ith coordinates are 0 and ith coordinate is 1. We have the following statement.

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Lemma 2.1. *Let A be a real positive definite symmetric $r \times r$ matrix. Then the system of equations*

$$1 - Q_i = \prod_{j=1}^{r} Q_j^{A_{ij}}, \qquad i = 1, \ldots, r$$

has a unique solution with $Q_i \in (0,1)$ for all $1 \le i \le r$.

Proof. We consider the function $f_A : [0,\infty)^r \to \mathbb{R}$ given by

$$f_A(x) = \frac{1}{2} x^T A x + \sum_{i=1}^{r} Li_2(\exp(-x_i)),$$

where Li_2 is the dilogarithm function defined by the power series $Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$ for $|z| < 1$. It has the property $zLi_2'(z) = -\log(1-z)$.

The gradient and the Hessian of f_A are

$$\nabla f_A(x) = Ax + (\log(1 - \exp(-x_i)))_{1 \le i \le r},$$
$$H_{f_A}(x) = A + \operatorname{diag}\left(\frac{1}{\exp(x_i) - 1}\right)_{1 \le i \le r}.$$

Using $Q_i = \exp(-x_i)$, the statement of the lemma is equivalent to saying that f_A has a unique critical point in $(0,\infty)^r$.

First, f_A has at least one critical point in $(0,\infty)^r$, because it takes on it's minimum in $(0,\infty)^r$: it's continuous, bounded from below by 0 and $f_A(x) \to \infty$ if $||x|| \to \infty$, and so it takes on it's minimum in $[0,\infty)^r$. In fact, it takes on that minimum in $(0,\infty)^r$, because

$$\lim_{x_i \downarrow 0} \frac{\partial f_A}{\partial x_i}(x) = -\infty < 0.$$

Second, f_A has at most one critical point in $(0,\infty)^r$, because it's differentiable and strictly convex on $(0,\infty)^r$: since A is positive definite, we see that the Hessian $H_{f_A}(x)$ is positive definite for all $x \in (0,\infty)^r$. □

Consider the unique solution $Q_i \in (0,1)$ of (1.2) and let $q = e^{-\varepsilon}$, $\varepsilon > 0$. Then one has

$$\frac{a_{n+e_i}(q)}{a_n(q)} \approx 1 \quad \forall i \quad \text{when} \quad n \approx \left(-\frac{\log Q_1}{\varepsilon}, \ldots, -\frac{\log Q_r}{\varepsilon}\right),$$

and it is very likely that $a_n(q)$ as a function of n is maximal around this point. We will apply a version of Laplace's method to describe the asymptotics of $F_{A,B,C}(e^{-\varepsilon})$ for small ε. For this we need the so called polylogarithm

$$Li_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m} \qquad \text{for} \quad |z| < 1, \; m \in \mathbb{Z},$$

which satisfies the obvious relation

$$z \frac{d}{dz} Li_m(z) = Li_{m-1}(z).$$

Lemma 2.2. *Let $n \in \mathbb{N}$ and $q = e^{-\varepsilon}$ with $\varepsilon > 0$. We fix $Q \in (0,1)$ and introduce a variable $\nu = -\log Q - n\varepsilon$. Then*

(i) for all n, ε we have an inequality

$$\log\Big(\frac{(q)_\infty}{(q)_n}\Big) \;<\; -\frac{Li_2(Q)}{\varepsilon} \;+\; \Big(\frac{\nu}{\varepsilon}-\frac{1}{2}\Big)\log(1-Q) \;+\; \frac{\nu}{2}\frac{Q}{1-Q}\,; \tag{2.1}$$

(ii) we have an asymptotic expansion

$$\log\Big(\frac{(q)_\infty}{(q)_n}\Big) \;\sim\; -\sum_{r,s\geq 0}\frac{Li_{2-r-s}(Q)B_r}{r!s!}\nu^s\varepsilon^{r-1} \quad \text{when } \varepsilon,\nu\to 0\,, \tag{2.2}$$

where $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, \ldots$ are the Bernoulli numbers.

Proof.

$$\begin{aligned}\log\Big(\frac{(q)_\infty}{(q)_n}\Big) \;&=\; \sum_{s=1}^{\infty}\log\big(1-q^{n+s}\big) \;=\; \sum_{s=1}^{\infty}\log\big(1-Qe^{\nu-s\varepsilon}\big)\\ &=\; -\sum_{s=1}^{\infty}\sum_{p=1}^{\infty}\frac{Q^pe^{p(\nu-s\varepsilon)}}{p} \;=\; -\sum_{p=1}^{\infty}\frac{Q^p}{p}\frac{e^{p\nu}}{e^{p\varepsilon}-1}\end{aligned}$$

Since $e^x > 1+x$ for all $x\neq 0$ and $\frac{x}{e^x-1} > 1+\frac{x}{2}$ for $x>0$ then

$$\frac{e^{p\nu}}{e^{p\varepsilon}-1} \;>\; (1+p\nu)\Big(\frac{1}{p\varepsilon}-\frac{1}{2}\Big) \;=\; \frac{1}{p\varepsilon}+\Big(\frac{\nu}{\varepsilon}-\frac{1}{2}\Big)-p\frac{\nu}{2}\,,$$

and we get inequality (i) after summation in p. To prove (ii) we notice that for every fixed p we have an asymptotic expansion

$$\frac{p\varepsilon e^{p\nu}}{e^{p\varepsilon}-1} \;\sim\; \Big(\sum_{r=0}^{\infty}\frac{B_r}{r!}(p\varepsilon)^r\Big)\Big(\sum_{s=0}^{\infty}\frac{(p\nu)^s}{s!}\Big) \;=\; \sum_{r,s\geq 0}\frac{B_r}{r!s!}(p\varepsilon)^r(p\nu)^s\,,$$

i.e. for every fixed N and $\delta>0$ we can find $\delta'>0$ such that

$$\frac{\Big|\frac{p\varepsilon e^{p\nu}}{e^{p\varepsilon}-1}-\sum_{r+s\leq N}\frac{B_rp^{r+s}}{r!s!}\varepsilon^r\nu^s\Big|}{p^N\max(\varepsilon,|\nu|)^N} < \delta$$

whenever $p\varepsilon, p|\nu| < \delta'$. Also we observe that when $x\searrow 0$

$$\frac{1}{x^N}\sum_{p>\frac{\delta'}{x}}p^aQ^p \to 0 \tag{2.3}$$

for any a, as well as

$$\frac{1}{x^N}\sum_{p>\frac{\delta'}{x}}\frac{Q^p}{p^2}\frac{p\varepsilon e^{p\nu}}{e^{p\varepsilon}-1} \;<\; \frac{1}{x^N}\sum_{p>\frac{\delta'}{x}}Q^pe^{p\nu} \;<\; \frac{1}{x^N}\frac{e^{\frac{\delta'}{x}(\nu+\log Q)}}{1-e^{\nu+\log Q}} \to 0 \tag{2.4}$$

uniformly in ν in small domains. Let us choose $\delta''>0$ such that expressions (2.3) for all integer a between -2 and $N-2$ and also the left-hand side of (2.4) are smaller than δ whenever $x<\delta''$ and $|\nu|<\delta''$. Now if

$\max(\varepsilon, |\nu|) < \delta''$ then

$$\begin{aligned}
&\frac{\left|\sum_{p\geq 1}\frac{Q^p}{p^2}\left(\frac{p\varepsilon e^{p\nu}}{e^{p\varepsilon}-1} - \sum_{r+s\leq N}\frac{B_r p^{r+s}}{r!s!}\varepsilon^r\nu^s\right)\right|}{\max(\varepsilon,|\nu|)^N} \leq \delta \sum_{p\max(\varepsilon,|\nu|)<\delta'} p^{N-2}Q^p \\
&\qquad + \frac{1}{\max(\varepsilon,|\nu|)^N}\sum_{p\max(\varepsilon,|\nu|)>\delta'}\frac{Q^p}{p^2}\frac{p\varepsilon e^{p\nu}}{e^{p\varepsilon}-1} \\
&\qquad + \sum_{r+s\leq N}\frac{|B_r|}{r!s!}\varepsilon^r|\nu|^s\frac{1}{\max(\varepsilon,|\nu|)^N}\sum_{p\max(\varepsilon,|\nu|)>\delta'} p^{r+s}Q^p \\
&\qquad \leq \Big(L_{2-N}(Q) + 1 + \sum_{r+s\leq N}\frac{|B_r|}{r!s!}(\delta'')^{r+s}\Big)\delta
\end{aligned}$$

and (ii) follows. □

Let $B_p(X) = \sum_k \binom{p}{k} B_k X^{p-k}$, $p \geq 1$ be the Bernoulli polynomials. Consider polynomials $D_p \in \mathbb{Q}[B, X, T]$, $p \geq 1$ defined by the following equality of formal power series in $\varepsilon^{1/2}$:

$$\begin{aligned}
&\exp\Big[(B + \frac{1}{2}\frac{Q}{1-Q})T\varepsilon^{1/2} - \sum_{p=3}^{\infty}\frac{1}{p!}B_p\big(\frac{T}{\varepsilon^{1/2}}\big)Li_{2-p}(Q)\,\varepsilon^{p-1}\Big] \\
&\qquad\qquad = 1 + \sum_{p=1}^{\infty} D_p\big(B, \frac{Q}{1-Q}, T\big)\,\varepsilon^{p/2}\,.
\end{aligned} \tag{2.5}$$

Observe that the coefficients of the series under the exponent are polynomials in B, $\frac{Q}{1-Q}$ and T because $Li_{2-r}(Q) = P_{r-1}\Big(\dfrac{Q}{1-Q}\Big)$ where P_r, $r \geq 1$ are the polynomials defined by $P_1(X) = X$ and $P_{p+1}(X) = (X^2 + X)\frac{d}{dX}P_p(X)$.

Theorem 2.3. *There is an asymptotic expansion*

$$F_{A,B,C}(e^{-\varepsilon})\,e^{-\frac{\alpha}{\varepsilon}} \sim \beta e^{-\gamma\varepsilon}\Big(1 + \sum_{p=1}^{\infty} c_p\varepsilon^p\Big), \quad \varepsilon \searrow 0$$

with the coefficients $\alpha \in \mathbb{R}_+$, $\beta, \gamma \in \overline{\mathbb{Q}}$ and $c_p \in \overline{\mathbb{Q}}$, $p \geq 1$ given below. Let $Q_i \in (0,1)$ be the solutions of (1.2). Denote $\xi_i = \frac{Q_i}{1-Q_i}$, $\widetilde{A} = A + \mathrm{diag}\{\xi_1, \ldots, \xi_r\}$ and let $L(x)$ be the Rogers dilogarithm function. Then

$$\begin{aligned}
\alpha &= \sum_{i=1}^{r}\big(L(1) - L(Q_i)\big) > 0\,, \\
\beta &= \det\widetilde{A}^{-1/2}\prod_i Q_i^{B_i}(1-Q_i)^{-1/2}\,, \quad \gamma = C + \frac{1}{24}\sum\frac{1+Q_i}{1-Q_i}\,, \\
c_p &= \det\widetilde{A}^{1/2}(2\pi)^{-r/2}\int C_{2p}(B,\xi,t)e^{-\frac{1}{2}t^T\widetilde{A}t}dt\,,
\end{aligned}$$

where the polynomials in $3r$ variables $C_p \in \mathbb{Q}[B,\xi,t]$ are defined as

$$C_p(B,\xi,t) = \sum_{p_1+\cdots+p_r=p} \prod_{i=1}^{p} D_{p_i}(B_i,\xi_i,t_i)\,,$$

where D_p are the polynomials in 3 variables defined by (2.5)*.*

Recall that $L(x)$ is an increasing function on $\mathbb{R}$ (therefore $\alpha > 0$), we have $L(x) = Li_2(x) + \frac{1}{2}\log(x)\log(1-x)$ for $x \in (0,1)$ and $L(1) = \frac{\pi^2}{6}$.

Proof. Let

$$\alpha' \;=\; -\sum_{i=1}^{r} L(Q_i)\,, \quad \beta' \;=\; \prod_i Q_i^{B_i}(1-Q_i)^{-1/2}\,, \quad \gamma' \;=\; C + \frac{1}{12}\sum_i \xi_i$$

and $t_i \;=\; -\frac{\log Q_i}{\varepsilon} - n_i$. Consider the function

$$\phi(t,\varepsilon) \;=\; \frac{(q)_\infty^r a_n(q)}{\beta' e^{\frac{\alpha'}{\varepsilon}}} \qquad (q = e^{-\varepsilon})$$

defined only for $t \in t^0(\varepsilon) + \mathbb{Z}^r$ where $t_i^0(\varepsilon)$ is the fractional part of $-\frac{\log Q_i}{\varepsilon}$. After a straightforward computation using (i) of Lemma 2.2 we obtain that

$$\log\phi(t,\varepsilon) \;<\; \Big(-\frac{1}{2}t^T At + t^T\big(B + \frac{1}{2}\xi\big) \;-\; C\Big)\varepsilon\,. \tag{2.6}$$

Then

$$\frac{(q)_\infty^r F_{A,B,C}(q)}{\beta' \exp(\frac{\alpha'}{\varepsilon})} \;=\; \sum_{t\in t^0+\mathbb{Z}^r} \phi(t,\varepsilon) \quad\sim\; \sum_{t\in t^0+\mathbb{Z}^r, |t_i|<\varepsilon^\lambda} \phi(t,\varepsilon)$$

for every $\lambda < -\frac{1}{2}$, where "$\sim$" always means that the difference is $o(\varepsilon^N)$ for every N. Indeed, for such λ we have $\sum_{|t_i|>\varepsilon^\lambda} \phi(t,\varepsilon) \;=\; o(\varepsilon^N)$ for every N due to (2.6). We can further rewrite it as

$$\sum_{t\in t^0+\mathbb{Z}^r, |t_i|<\varepsilon^\lambda} \phi(t,\varepsilon) \quad=\; \sum_{t\in (t^0+\mathbb{Z}^r)\sqrt{\varepsilon}, |t_i|<\varepsilon^{\lambda+\frac{1}{2}}} \phi\big(\frac{t}{\sqrt{\varepsilon}},\varepsilon\big)\,.$$

Let also $\lambda > -\frac{2}{3}$. Then

$$\phi\big(\frac{t}{\sqrt{\varepsilon}},\varepsilon\big) \;=\; e^{-\frac{1}{2}t^t\widetilde{A}t - \gamma'\varepsilon}\Big(1 + \sum_{p=1}^{N} C_p(t)\varepsilon^{p/2}\Big) \;+\; o(\varepsilon^{N(3\lambda+2)}) \tag{2.7}$$

uniformly in the domain $|t_i| \le \varepsilon^{\lambda+\frac{1}{2}}$, and we observe that for any polynomial P

$$\sum_{t\in (t^0+\mathbb{Z}^r)\sqrt{\varepsilon}, |t_i|<\varepsilon^{\lambda+\frac{1}{2}}} P(t)e^{-\frac{1}{2}t^T\widetilde{A}t} \;\sim\; \varepsilon^{-r/2}\int P(t)e^{-\frac{1}{2}t^T\widetilde{A}t}dt \tag{2.8}$$

(the difference is $o(\varepsilon^N)$ for every N) when $\lambda < -\frac{1}{2}$. Combining (2.7) and (2.8) (we will prove both facts later), we get

$$\frac{(q)_\infty^r F_{A,B,C}(q)}{\beta' \exp(\frac{\alpha'}{\varepsilon} - \gamma'\varepsilon)} \;\sim\; \Big(\frac{2\pi}{\varepsilon}\Big)^{r/2} \det\widetilde{A}^{-1/2}\Big(1 + \sum_{p=1}^{\infty} c_p\varepsilon^p\Big)\,.$$

Here we have only integer powers of ε because $\int C_p(t)e^{-\frac{1}{2}t^T\widetilde{A}t}dt = 0$ when p is odd. And this happens because the total t-degree of every monomial in C_p has the same parity as p, which in turn follows from the definition of D_p. Now, since

$$\log(q)_\infty \;\sim\; -\frac{\pi^2}{6}\frac{1}{\varepsilon} \;+\; \frac{1}{2}\log\Big(\frac{2\pi}{\varepsilon}\Big) \;+\; \frac{\varepsilon}{24}$$

when $\varepsilon \to 0$, we obtain the statement of the theorem.

To prove (2.8) we notice again that $\sum_{|t_i|>\varepsilon^{\lambda+\frac{1}{2}}} P(t)e^{-\frac{1}{2}t^t\widetilde{A}t} \;=\; o(\varepsilon^N)$ for every N, and using Poisson summation formula we have

$$\sum_{t\in(t^0+\mathbb{Z}^r)\sqrt{\varepsilon}} P(t)e^{-\frac{1}{2}t^T\widetilde{A}t} \;=\; \sum_{t\in t^0+\mathbb{Z}^r} P(t\sqrt{\varepsilon})e^{-\frac{\varepsilon}{2}t^T\widetilde{A}t} \;=\; \sum_{s\in\mathbb{Z}^r} g(s)e^{2\pi i s^T t^0}\,,$$

where $g(s)$ is the Fourier transform of $P(t\sqrt{\varepsilon})e^{-\frac{\varepsilon}{2}t^T\widetilde{A}t}$. Then $g(0)$ is the right-hand side of (2.8), and the sum of all remaining terms are $o(\varepsilon^N)$ since for any monomial $P'(t)$ and $g'(s)$ being the Fourier transform of $P'(t)e^{-\frac{\varepsilon}{2}t^tA't}$ one can check by direct computation that $\sum_{s\in\mathbb{Z}^2\setminus\{0\}}|g'(s)| = o(\varepsilon^N)$ for any N.

It remains to prove (2.7). Using (ii) of Lemma 2.2 we get

$$\begin{aligned}\log\phi(t,\varepsilon) \;\sim\; & -\frac{1}{2}t^T\widetilde{A}t - \gamma'\varepsilon + t^T\big(B+\frac{1}{2}\xi\big)\\ & -\sum_i\sum_{p=3}^{\infty}\frac{1}{p!}B_p(t_i)Li_{2-p}(Q_i)\varepsilon^{p-1}\,, \qquad \varepsilon, t\varepsilon\to 0\,,\end{aligned}$$

and therefore for every N

$$\begin{aligned}\log\phi\big(\frac{t}{\sqrt{\varepsilon}},\varepsilon\big) \;=\; & -\frac{1}{2}t^T\widetilde{A}t - \gamma'\varepsilon + t^T\big(B+\frac{1}{2}\xi\big)\sqrt{\varepsilon}\\ & -\sum_i\sum_{p=3}^{N}\frac{1}{p!}B_p\Big(\frac{t_i}{\sqrt{\varepsilon}}\Big)Li_{2-p}(Q_i)\varepsilon^{p-1} \;+\; o(\varepsilon^{N(\lambda+1)-1})\end{aligned}$$

uniformly in $|t_i|\le\varepsilon^{\lambda+\frac{1}{2}}$. If we rewrite the right-hand side as $\sum_{p=0}^{N-2}g_p(t)\varepsilon^{\frac{p}{2}}$ then $\deg g_p \le p+2$ (because $\deg B_p = p$). It follows that $\sum_{p=1}^{N-2}g_p(t)\varepsilon^{\frac{p}{2}} = O(\varepsilon^{3\lambda+2})$ uniformly in our domain since

$$(\lambda+\frac{1}{2})(p+2) \;+\; \frac{p}{2} \;=\; p(\lambda+1) \;+\; 2\lambda \;+\; 1 \;\ge\; 3\lambda \;+\; 2 \;>\; 0\,.$$

Therefore we can take a sufficiently long but finite part of the standard series to approximate its exponent. Hence some sufficiently long but again finite part of

$$\exp\Big[\sum_i(B_i+\frac{1}{2}\xi)t_i\sqrt{\varepsilon} - \sum_i\sum_{p=3}^{\infty}\frac{1}{p!}B_p\big(\frac{t_i}{\sqrt{\varepsilon}}\big)Li_{2-p}(Q_i)\varepsilon^{p-1}\Big] \;=\; 1+\sum_{p=1}^{\infty}C_p(t)\varepsilon^{p/2}$$

will approximate $\phi\big(\frac{t}{\sqrt{\varepsilon}},\varepsilon\big)e^{\frac{1}{2}t^T\widetilde{A}t+\gamma'\varepsilon}$. One can easily see that $\deg C_p(t)\le 3p$ (in the variable t). Since for $p>N$

$$C_p(t)\varepsilon^{\frac{p}{2}} \;=\; O(\varepsilon^{(\lambda+\frac{1}{2})3p+\frac{p}{2}}) \;=\; O(\varepsilon^{p(3\lambda+2)}) \;=\; o(\varepsilon^{N(3\lambda+2)})$$

then it is sufficient to consider only the part with $p \le N$ in (2.7). $\square$

3. Modular functions $F_{A,B,C}$

Let us search for those triples (A,B,C) for which $F_{A,B,C}(q)$ is a modular function (of any weight and any congruence subgroup). We will call such (A,B,C) *a modular triple.* The idea here is that in order for $F_{A,B,C}(q)$ to be modular, the asymptotic expansion needs to be of a special type, as we can see from the following lemma.

Lemma 3.1. *Let $F(q)$ be a modular form of weight w for some subgroup of finite index $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$. Then for some numbers $a \in \pi^2\mathbb{Q}$ and $b \in \mathbb{C}$*

$$e^{\frac{a}{\varepsilon}} F(e^{-\varepsilon}) \;\sim\; b\,\varepsilon^{-w} \;+\; o(\varepsilon^N) \qquad \forall N \ge 0\,. \tag{3.1}$$

Proof. The group $\mathrm{SL}(2,\mathbb{Z})$ acts on the space $M_w(\Gamma)$ of modular functions of weight w. Let $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $SF \in M_w(\Gamma)$, and in particular it has a q-expansion (with some rational powers of $q = e^{2\pi i z}$):

$$\frac{1}{z^w} F\Big(e^{-2\pi i \frac{1}{z}}\Big) \;=\; a_0 q^{\alpha_0} \;+\; a_1 q^{\alpha_1} \;+\; \dots$$

We subsitute $z = \frac{2\pi i}{\varepsilon}$ and get

$$\begin{aligned} F(e^{-\varepsilon}) \;&=\; \Big(\frac{2\pi i}{\varepsilon}\Big)^w \Big[a_0 e^{-\frac{4\pi^2\alpha_0}{\varepsilon}} \;+\; a_1 e^{-\frac{4\pi^2\alpha_1}{\varepsilon}} \;+\; \dots\Big] \\ &=\; \frac{(2\pi i)^w a_0}{\varepsilon^w} e^{-\frac{4\pi^2\alpha_0}{\varepsilon}} \Big[1 \;+\; o(\varepsilon^N)\Big] \qquad \forall N\,. \end{aligned}$$

$\square$

If we now compare the asymptotics from Theorem 2.3 with (3.1) we get the following statement.

Corollary 3.2. *If $F_{A,B,C}(q)$ is modular then*

(i) *its weight $w = 0$*
(ii) $\alpha \in \pi^2\mathbb{Q} \iff \sum_{i=1}^r L(Q_i) \in \pi^2\mathbb{Q}$
(iii) $e^{-\gamma\varepsilon}\big(1 + \sum_{p=1}^\infty c_p \varepsilon^p\big) = 1 \iff c_p = \frac{\gamma^p}{p!} \quad \forall p$

Condition (ii) is very interesting, we consider it in the next section. It follows from (iii) that modular triples satisfy an infinite number of equations

$$\big(c_p \;-\; \frac{1}{p!} c_1^p\big)\big(B, \xi, \widetilde{A}^{-1}\big) \;=\; 0\,, \quad p = 2, 3, \dots\,, \tag{3.2}$$

and these equations are polynomial in the entries of $B, \xi, \widetilde{A}^{-1}$. Indeed, let us look at the expression for c_p from Theorem 2.3. Since the generating function for the moments of the Gaussian measure is

$$\sum_{a \in (\mathbb{Z}_{\ge 0})^r} \frac{x^a}{a_1! \dots a_r!} \frac{\det \widetilde{A}^{1/2}}{(2\pi)^{r/2}} \int t^a e^{-\frac{1}{2} t^T \widetilde{A} t} dt \;=\; \exp\big(\frac{1}{2} x^T \widetilde{A}^{-1} x\big)\,,$$

all the moments are rational polynomials in the entries of $\widetilde{A}^{-1}$ and we obtain that $c_p \in \mathbb{Q}[B, \xi, \widetilde{A}^{-1}]$.

Now let $r = 1$. It is easy to see that the degrees of $D_p(B, X, T)$ in the variables B, X and T are $p, 2p$ and $3p$, respectively. Since $c_p(B, \xi, (A+\xi)^{-1})$ is the integral of $D_{2p}(B, \xi, t)$ w.r.t. the measure $\frac{(A+\xi)^{1/2}}{\sqrt{2\pi}} e^{-(A+\xi)t^2/2} dt$ and the integral of t^{2m} is $(2m-1)!!(A+\xi)^{-m}$, the degrees of c_p in the corresponding variables are $2p$, $4p$ and $3p$. It is convenient to consider the polynomials

$$\widetilde{c}_p(B, \xi, A) = (A+\xi)^{3p} \big[c_p - \frac{1}{p!} c_1^p\big](B, \xi, \frac{1}{A+\xi}), \qquad p = 2, 3, \ldots$$

Although these polynomials look rather complicated, we have found using the *Magma algebra system* ([1]) that the ideal

$$I = \langle \widetilde{c}_2, \widetilde{c}_3, \widetilde{c}_4, \widetilde{c}_5 \rangle \subset \mathbb{Q}[B, \xi, A]$$

contains the element

$$\xi(\xi+1)A^{13}(A-1)^{13}(A+1)(A-2)(A-1/2)\,.$$

Consequently, if (A, B, C) is a modular triple then $A \in \{\frac{1}{2}, 1, 2\}$. For each A on this list it is not hard to find the corresponding values of B, and one can compute C from the equality $\gamma = c_1$. This way we obtain exactly the list from the theorem below.

Theorem 3.3 (D. Zagier [8]). *Let $r = 1$. The only $(A, B, C) \in \mathbb{Q}_+ \times \mathbb{Q} \times \mathbb{Q}$ for which $F_{A,B,C}(q)$ is a modular form are given in the following table.*

A	B	C	$F_{A,B,C}(e^{2\pi i z})$
2	0	$-1/60$	$\theta_{5,1}(z)/\eta(z)$
	1	$11/60$	$\theta_{5,2}(z)/\eta(z)$
1	0	$-1/48$	$\eta(z)^2/\eta(\frac{z}{2})\eta(2z)$
	$1/2$	$1/24$	$\eta(2z)/\eta(z)$
	$-1/2$	$1/24$	$2\eta(2z)/\eta(z)$
$1/2$	0	$-1/40$	$\theta_{5,1}(\frac{z}{4})\eta(2z)/\eta(z)\eta(4z)$
	$1/2$	$1/40$	$\theta_{5,2}(\frac{z}{4})\eta(2z)/\eta(z)\eta(4z)$

Here $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ and $\theta_{5,j}(z) = \sum_{n\equiv 2j-1(10)}(-1)^{[n/10]}q^{n^2/40}$.

We warn the reader that if (iii) of Corollary 3.2 holds for some (A, B, C) this does not yet imply that $F_{A,B,C}$ is in fact modular. To get modularity one needs to prove an identity between the corresponding q-series for each line of the table. For example, the first two lines correspond to the well known Rogers–Ramanujan identities.

Further computer experiments showed that $\widetilde{c}_p \in I$ for $p = 6, \ldots, 20$. Although we stopped at this point, it is very likely that the statement is true for all p. Also with the help of *Magma* we have got the following decomposition of the radical of I into prime ideals:

$$\mathrm{Rad}(I) \;=\; \mathcal{P}_1 \cdot \ldots \cdot \mathcal{P}_{14}$$

where the generators of $\mathcal{P}_i$ are given below:

i	generators of $\mathcal{P}_i$		
1		ξ	
2		$\xi+1$	
3	$B-1/2$,	$\xi+2$,	A
4	$B-1$,	$\xi+2$,	A
5	B,	$\xi+2$,	A
6	$B+1/2$,	$\xi^2+3\xi+1$,	$A+1$
7	$B-1/2$,	$\xi^2+3\xi+1$,	$A+1$
8	$B+1/2$,	$\xi-1$,	$A-1$
9	B,	$\xi-1$,	$A-1$
10	$B-1/2$,	$\xi-1$,	$A-1$
11	$B-1$,	$\xi^2-\xi-1$,	$A-2$
12	B,	$\xi^2-\xi-1$,	$A-2$
13	$B-1/2$,	$\xi^2+\xi-1$,	$A-1/2$
14	B,	$\xi^2+\xi-1$,	$A-1/2$

Consequently, the set of all solutions of the system $\widetilde{c}_p(B,\xi,A)=0$, $p=2,3,\ldots$ is a subset of this table, and if we indeed had $\widetilde{c}_p\in I$ (or at least $\widetilde{c}_p\in\mathrm{Rad}(I)$) for all p then this table would be exactly the set of solutions.

Let's us now consider the case $r=2$. The task of solving the system (3.2) for several small values of p becomes already very complicated. We failed to solve it with *Magma* in full generality for $r=2$ as we did in the case $r=1$. However, we can still search for modular $F_{A,B,C}$, where A is of a special type. We will consider three families of matrices:

$$
\begin{aligned}
A &= \begin{pmatrix} a & \frac{1}{2}-a \\ \frac{1}{2}-a & a \end{pmatrix} \quad \Rightarrow \quad \xi_1=\xi_2=\frac{\sqrt{5}-1}{2}\\
A &= \begin{pmatrix} a & 2-a \\ 2-a & a \end{pmatrix} \quad \Rightarrow \quad \xi_1=\xi_2=\frac{\sqrt{5}+1}{2}\\
A &= \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix} \quad \Rightarrow \quad \xi_1=\xi_2=\frac{1}{2}
\end{aligned}
\tag{3.3}
$$

It is easy to check that (ii) of Corollary 3.2 holds for these matrices. For these families of matrices we can do an analysis similar to what we did for $r=1$.

Theorem 3.4. *Modular functions $F_{A,B,C}(z)$ with the matrix A being of the form $\begin{pmatrix} a & \frac{1}{2}-a \\ \frac{1}{2}-a & a \end{pmatrix}$ exist if and only if $a=1$, $a=3/4$ or $a=1/2$. Below is the list of all such modular functions.*

A	B	C	$F_{A,B,C}(e^{2\pi i z})$
$\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$-\frac{1}{20}$	$(\theta_{5,\frac{3}{4}}(2z) + \theta_{5,\frac{13}{4}}(2z))\eta(z)/\eta(2z)\eta(z/2) + 2\theta_{5,2}(2z)\eta(2z)/\eta(z)^2$
	$\begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$	$\frac{1}{20}$	$2\theta_{5,1}(2z)\eta(2z)/\eta(z)^2 + \theta_{5,\frac{3}{2}}(z)\theta_{5,2}(2z)\eta(z)^3/\eta(z/2)^2\eta(2z)^2\eta(10z)$
$\begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$	$-\frac{1}{80}$	$\theta_{5,1}(\frac{z}{8})\eta(z)/\eta(\frac{z}{2})\eta(2z)$
	$\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$	$\frac{1}{80}$	$\theta_{5,2}(\frac{z}{8})\eta(z)/\eta(\frac{z}{2})\eta(2z)$
$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$-\frac{1}{20}$	$(\theta_{5,1}(\frac{z}{4})\eta(2z)/\eta(z)\eta(4z))^2$
	$\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$	0	$\theta_{5,1}(\frac{z}{4})\theta_{5,2}(\frac{z}{4})(\eta(2z)/\eta(z)\eta(4z))^2$
	$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$	$\frac{1}{20}$	$(\theta_{5,2}(\frac{z}{4})\eta(2z)/\eta(z)\eta(4z))^2$

Proof. Consider the ideal $I \subset \mathbb{Q}[b_1, b_2, \xi, a]$ generated by $\xi^2 + \xi - 1$ and the polynomials

$$(\xi^2 + 2a\xi + a - 1/4)^{3p} \times \left[c_p - \frac{c_1^p}{p!}\right]\left(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \begin{pmatrix} a+\xi & \frac{1}{2}-a \\ \frac{1}{2}-a & a+\xi \end{pmatrix}^{-1}\right)$$

for $p = 2, 3, 4, 5$. We find with *Magma* that the element

$$a(a - \frac{1}{4})(a - \frac{1}{2})(a - \frac{3}{4})(a-1)(a^2 - a - \frac{1}{16})$$

belongs to I. (We ran the function *GroebnerBasis(I)* which has computed the Groebner basis for I using reversed lexicografical order on monomials with the variables ordered as $b_1 > b_2 > \xi > a$. It took several hours, the Groebner basis contains 15 elements, and the element above is one of them.) The last term doesn't give rational values for a, and the reason it enters here is that we have multiplied every equation $c_p - c_1^p/p! = 0$ by $3p$th power of the determinant $\xi^2 + 2a\xi + a - 1/4 = (\xi + 1/2)(\xi + 2a - 1/2)$ while precisely the denominator of $c_p - c_1^p/p!$ is $(\xi + 1/2)^{3p}(\xi + 2a - 1/2)^{2p}$. Therefore our polynomials are divisible by $(\xi + 2a - \frac{1}{2})^p$ for $p = 2, 3, 4, 5$, and since $\xi^2 + \xi = 1$ these factors are zero exactly when $a^2 - a = \frac{1}{16}$. We now have a finite list of values for a, and we plug each of them together with ξ into the equations to find all values of b_1 and b_2 for which our equations vanish for $p = 2, 3, 4, 5$. So, we get the list above. For each row we compute the corresponding value of C from $c_1 = \gamma$, i.e.

$$C \;=\; c_1(b_1, b_2, \xi, a) - \frac{1}{24}\sum_i \frac{1+Q_i}{1-Q_i} \;=\; c_1(b_1, b_2, \xi, a) - \frac{2\xi + 1}{12}\,.$$

What remains is to prove that the $F_{A,B,C}$ satisfy the identities given in the last column. For the case $a = 1/2$, this is easy, since $F_{A,B,C}$ splits as the product of two rank 1 cases, for which an identity is given in Theorem 3.3. For the case $a = 3/4$, the identities follow directly by applying Theorem 4.2 below, with $m = 2$ and $A = 1/2$, and again using identities from Theorem 3.3.

Only the case $a = 1$ is a bit more work: using

$$(-xq^{1/2};q)_\infty = \sum_{k\geq 0} \frac{q^{\frac{1}{2}k^2}x^k}{(q)_k} \tag{3.4}$$

(this is a direct consequence of (7) in Chapter 2 of [8]), with $x = q^{-n/2}$, we find

$$\begin{aligned}\sum_{m,n\geq 0} \frac{q^{\frac{1}{2}m^2-\frac{1}{2}mn+\frac{1}{2}n^2}}{(q)_m(q)_n} &= \sum_{n\geq 0} \frac{q^{\frac{1}{2}n^2}(-q^{-\frac{1}{2}n+\frac{1}{2}})_\infty}{(q)_n} \\ &= \sum_{n\geq 0} \frac{q^{2n^2}(-q^{-n+\frac{1}{2}})_\infty}{(q)_{2n}} + \sum_{n\geq 0} \frac{q^{2n^2+2n+\frac{1}{2}}(-q^{-n})_\infty}{(q)_{2n+1}}.\end{aligned}$$

Now using that for $n \geq 0$ we have $(-q^{-n+\frac{1}{2}})_\infty = q^{-\frac{1}{2}n^2}(-q^{\frac{1}{2}})_n(-q^{\frac{1}{2}})_\infty$ and $(-q^{-n})_\infty = 2q^{-\frac{1}{2}n^2-\frac{1}{2}n}(-q)_n(-q)_\infty$, this equals

$$\begin{aligned}&(-q^{\frac{1}{2}})_\infty \sum_{n\geq 0} \frac{q^{\frac{3}{2}n^2}(-q^{\frac{1}{2}})_n}{(q)_{2n}} + 2q^{\frac{1}{2}}(-q)_\infty \sum_{n\geq 0} \frac{q^{\frac{3}{2}n^2+\frac{3}{2}n}(-q)_n}{(q)_{2n+1}} \\ =&(-q^{\frac{1}{2}})_\infty \sum_{n\geq 0} \frac{q^{\frac{3}{2}n^2}}{(q^{\frac{1}{2}})_n(q^2;q^2)_n} + 2q^{\frac{1}{2}}(-q)_\infty \sum_{n\geq 0} \frac{q^{\frac{3}{2}n^2+\frac{3}{2}n}}{(q)_n(q;q^2)_{n+1}}.\end{aligned} \tag{3.5}$$

To get identities for these last two sums, we use equations (19) and (44) in [6], which (in our notation) read

$$\begin{aligned}\sum_{n\geq 0} \frac{(-1)^n q^{3n^2}}{(-q;q^2)_n(q^4;q^4)_n} &= \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty}{(q^2;q^2)_\infty}, \\ \sum_{n\geq 0} \frac{q^{\frac{3}{2}n^2+\frac{3}{2}n}}{(q)_n(q;q^2)_{n+1}} &= \frac{(q^2;q^{10})_\infty(q^8;q^{10})_\infty(q^{10};q^{10})_\infty}{(q)_\infty}.\end{aligned}$$

If we use the Jacobi triple product identity $(-xq^{1/2})_\infty(-x^{-1}q^{1/2})_\infty(q)_\infty = \sum_{n\in\mathbb{Z}} x^n q^{n^2/2}$ on the right hand sides and replace q by $-q^{1/2}$ in the first identity, we get

$$\begin{aligned}\sum_{n\geq 0} \frac{q^{\frac{3}{2}n^2}}{(q^{\frac{1}{2}})_n(q^2;q^2)_n} &= q^{\frac{7}{120}} \frac{\theta_{5,\frac{3}{4}}(2z) + \theta_{5,\frac{13}{4}}(2z)}{\eta(z)}, \\ \sum_{n\geq 0} \frac{q^{\frac{3}{2}n^2+\frac{3}{2}n}}{(q)_n(q;q^2)_{n+1}} &= q^{-\frac{49}{120}} \frac{\theta_{5,2}(2z)}{\eta(z)}.\end{aligned}$$

Further we have

$$(-q^{\frac{1}{2}})_\infty = \frac{(q;q^2)_\infty}{(q^{\frac{1}{2}};q)_\infty} = \frac{(q)_\infty^2}{(q^2;q^2)_\infty(q^{\frac{1}{2}};q^{\frac{1}{2}})_\infty} = q^{\frac{1}{48}}\frac{\eta(z)^2}{\eta(2z)\eta(z/2)},$$

$$(-q)_\infty = \frac{(q^2;q^2)_\infty}{(q)_\infty} = q^{-\frac{1}{24}}\frac{\eta(2z)}{\eta(z)},$$

and so we get from (3.5)

$$F_{A,B,C}(q) = \frac{\eta(z)}{\eta(2z)\eta(z/2)}\left(\theta_{5,\frac{3}{4}}(2z)+\theta_{5,\frac{13}{4}}(2z)\right) + 2\frac{\eta(2z)}{\eta(z)^2}\theta_{5,2}(2z),$$

where $A = \left(\begin{smallmatrix} 1 & -1/2 \\ -1/2 & 1\end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 0 \\ 0\end{smallmatrix}\right)$ and $C = -1/20$.

The proof for the identity for $B = \left(\begin{smallmatrix} -1/2 \\ 0\end{smallmatrix}\right)$ and $C = 1/20$ is very similar, and so we omit some of the details. We have

$$\begin{aligned}
&\sum_{m,n\geq 0} \frac{q^{\frac{1}{2}m^2-\frac{1}{2}mn+\frac{1}{2}n^2-\frac{1}{2}m}}{(q)_m(q)_n} \\
&\qquad = 2(-q)_\infty \sum_{n\geq 0}\frac{q^{\frac{3}{2}n^2-\frac{1}{2}n}(-q)_n}{(q)_{2n}} + (-q^{\frac{1}{2}})_\infty\sum_{n\geq 0}\frac{q^{\frac{3}{2}n^2+n}(-q^{\frac{1}{2}})_{n+1}}{(q)_{2n+1}} \\
&\qquad = 2(-q)_\infty \sum_{n\geq 0}\frac{q^{\frac{3}{2}n^2-\frac{1}{2}n}}{(q)_n(q;q^2)_n} + (-q^{\frac{1}{2}})_\infty\sum_{n\geq 0}\frac{q^{\frac{3}{2}n^2+n}(-q^{\frac{1}{2}})_{n+1}}{(q)_{2n+1}}.
\end{aligned}$$

Again we use two identities from Slater's list (see [6]), namely (46) which reads

$$\sum_{n\geq 0}\frac{q^{\frac{3}{2}n^2-\frac{1}{2}n}}{(q)_n(q;q^2)_n} = \frac{(q^4;q^{10})_\infty(q^6;q^{10})_\infty(q^{10};q^{10})_\infty}{(q)_\infty},$$

and so we can identify it as $q^{-1/120}\theta_{5,1}(2z)/\eta(z)$, and (97), which should read (note that there are mistakes in some of the exponents; we have given the corrected version here)

$$\begin{aligned}
&\sum_{n\geq 0}\frac{q^{3n^2+2n}(-q;q^2)_{n+1}}{(q^2;q^2)_{2n+1}} \\
&\quad = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left((-q^{11};q^{30})_\infty(-q^{19};q^{30})_\infty - q^3(-q;q^{30})_\infty(-q^{29};q^{30})_\infty\right)(q^{30};q^{30})_\infty \\
&\quad = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q^3;q^{10})_\infty(q^7;q^{10})_\infty(q^{10};q^{10})_\infty(q^4;q^{20})_\infty(q^{16};q^{20})_\infty.
\end{aligned}$$

If we replace q by $q^{1/2}$, we find

$$\sum_{n\geq 0}\frac{q^{\frac{3}{2}n^2+n}(-q^{\frac{1}{2}})_{n+1}}{(q)_{2n+1}} = q^{-\frac{17}{240}}\frac{\theta_{5,\frac{3}{2}}(z)\theta_{5,2}(2z)\eta(z)}{\eta(z/2)\eta(2z)\eta(10z)},$$

which gives the desired result. □

In [8] one can find a list of triples (A,B,C) for $r=2$ (Table 2 on p. 47) for which numerical experiments show that the condition (iii) of Corollary 3.2 holds, as well as (ii). We see that the cases of Theorem 3.4 with $a=1$ are

on this list, but the ones with $a = 3/4$ appear to be new. We will come back to the case $a = 3/4$ in the next section.

Similar analysis for the other two families in (3.3) gave the following results. In both cases if the matrix in the family is diagonal then the modular forms are products of the ones from Theorem 3.3. Non-diagonal cases are listed in Tables 1 and 2.

TABLE 1. A complete list of modular triples (A, B, C) with the matrix

$$A = \begin{pmatrix} a & 2-a \\ 2-a & a \end{pmatrix},\ a > 1,\ a \neq 2.$$

A	B	C	$F_{A,B,C}(e^{2\pi i z})$
$\begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$-\frac{1}{30}$	? $\frac{1}{\eta(z)} \sum_{n\in\mathbb{Z}} (-1)^n \big(2q^{\frac{15}{2}(n+\frac{3}{10})^2} + q^{\frac{15}{2}(n+\frac{1}{30})^2} - q^{\frac{15}{2}(n+\frac{11}{30})^2}\big)$
	$\begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$	$\frac{1}{30}$	? $\frac{1}{\eta(z)} \sum_{n\in\mathbb{Z}} (-1)^n \big(2q^{\frac{15}{2}(n+\frac{1}{10})^2} + q^{\frac{15}{2}(n+\frac{13}{30})^2} - q^{\frac{15}{2}(n+\frac{23}{30})^2}\big)$
$\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$	$-\frac{1}{120}$	$\theta_{5,1}(\frac{z}{2})/\eta(\frac{z}{2})$
	$\begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}$ and $\begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}$	$\frac{11}{120}$	$\theta_{5,2}(\frac{z}{2})/\eta(\frac{z}{2})$

In Table 1, the identities for the case $a = 3/2$ follow directly by applying Theorem 4.2 with $m = 2$ and $A = 2$, and using identities from Theorem 3.3. For the case $a = 4/3$ we were unable to find a proof, but we verified them to a high order in the power series in q.

In Table 2, the identity for $B = \left(\begin{smallmatrix} b \\ -b \end{smallmatrix}\right)$ is given in [8] (see (26) in Chapter 2). The proof uses that for any $n \in \mathbb{Z}$

$$\sum_{\substack{k,l\geq 0 \\ k-l=n}} \frac{q^{kl}}{(q)_k (q)_l} = \frac{1}{(q)_\infty}. \tag{3.6}$$

The identity for $B = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$ is proven similarly, using

$$\sum_{\substack{k,l\geq 0 \\ k-l=n}} \frac{q^{kl-\frac{1}{2}k-\frac{1}{2}l}}{(q)_k (q)_l} = \frac{q^{n/2} + q^{-n/2}}{(q)_\infty},$$

TABLE 2. The list containing all (B,C) such that $F_{A,B,C}$ is modular, where

$$A=\begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix},\ a>\tfrac{1}{2},\ a\neq 1.$$

B	C	$F_{A,B,C}(e^{2\pi iz})$
$\begin{pmatrix} b \\ -b \end{pmatrix}$	$\frac{b^2}{2a}-\frac{1}{24}$	$\frac{1}{\eta(z)}\sum_{n\in\mathbb{Z}+\frac{b}{a}} q^{an^2/2}$
$\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$\frac{1}{8a}-\frac{1}{24}$	$\frac{2}{\eta(z)}\sum_{n\in\mathbb{Z}+\frac{1}{2a}} q^{an^2/2}$
$\begin{pmatrix} 1-\frac{a}{2} \\ \frac{a}{2} \end{pmatrix}$ and $\begin{pmatrix} \frac{a}{2} \\ 1-\frac{a}{2} \end{pmatrix}$	$\frac{a}{8}-\frac{1}{24}$	$\frac{1}{2\eta(z)}\sum_{n\in\mathbb{Z}+\frac{1}{2}} q^{an^2/2}$

for all $n\in\mathbb{Z}$. This identity follows directly from (3.6):

$$\begin{aligned}\sum_{\substack{k,l\geq 0\\ k-l=n}} \frac{q^{kl-\frac{1}{2}k-\frac{1}{2}l}}{(q)_k(q)_l} &= \sum_{\substack{k,l\geq 0\\ k-l=n}} \frac{q^{kl-\frac{1}{2}k-\frac{1}{2}l}\left((1-q^k)+q^k\right)}{(q)_k(q)_l}\\ &= \sum_{\substack{k\geq 1,l\geq 0\\ k-l=n}} \frac{q^{kl-\frac{1}{2}k-\frac{1}{2}l}}{(q)_{k-1}(q)_l} + \sum_{\substack{k,l\geq 0\\ k-l=n}} \frac{q^{kl+\frac{1}{2}k-\frac{1}{2}l}}{(q)_k(q)_l}.\end{aligned}$$

If we replace k by $k+1$ in the first sum on the RHS, we see that it equals $q^{-n/2}/(q)_\infty$ and the second sum equals $q^{n/2}/(q)_\infty$.

To get the identity for $B=\begin{pmatrix} \frac{a}{2} \\ 1-\frac{a}{2} \end{pmatrix}$ we use

$$\sum_{\substack{k,l\geq 0\\ k-l=n}} \frac{q^{kl+l}}{(q)_k(q)_l} = \frac{1}{(q)_\infty}(-1)^n q^{-\frac{1}{2}n^2-\frac{1}{2}n} s_n, \tag{3.7}$$

with $s_n=\sum_{k\geq n}(-1)^k q^{\frac{1}{2}k^2+\frac{1}{2}k}$ (this is easily obtained by checking that both sides satisfy the recursion $b_n+q^{n+1}b_{n+1}=1/(q)_\infty$ and $\lim_{n\to\infty} b_n=\frac{1}{(q)_\infty}$) to get

$$F_{A,B,C}(q)=\frac{q^{a/8}}{\eta(z)}\sum_{n\in\mathbb{Z}} q^{(a-1)(n^2+n)/2} s_n.$$

If we replace n by $-n-1$ in the sum and use that $s_{-n-1}=s_{n+1}=s_n-(-1)^n q^{\frac{1}{2}n^2+\frac{1}{2}n}$, we easily get that $\sum_{n\in\mathbb{Z}} q^{(a-1)(n^2+n)/2}s_n=\frac{1}{2}\sum_{n\in\mathbb{Z}} q^{a(n^2+n)/2}$, which gives the desired result.

We also checked for each matrix A in Zagier's list for $r=2$ (p. 47 in [8]) if the corresponding list of vectors B is complete. It appears to be complete in

all cases except $A = \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}$. For such matrices only the modular forms in the first row of Table 2 were known.

4. Counterexamples to Nahm's conjecture

The Bloch group $B(K)$ of a field K is an abelian group defined as the quotient of the kernel of the map

$$\begin{aligned} \mathbb{Z}[K^* \setminus 1] &\to \Lambda^2 K^* \\ x &\mapsto x \wedge (1-x) \end{aligned} \tag{4.1}$$

by the subgroup generated by all elements of the form

$$[x] + [1-x]\,, \quad [x] + [\frac{1}{x}]\,, \quad [x]+[y]+[1-xy]+[\frac{1-x}{1-xy}]+[\frac{1-y}{1-xy}]\,. \tag{4.2}$$

If K is a number field than $B(K) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K_3(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ and the regulator map is given explicitly on $B(K)$ by

$$\begin{aligned} B(K) &\to \mathbb{R}^{r_2} \\ x &\mapsto (D(\sigma_1(x)), \dots, D(\sigma_{r_2}(x))) \end{aligned}$$

where r_2 is the number of pairs of complex conjugate embeddings of K into $\mathbb{C}$, $\sigma_1, \dots, \sigma_{r_2}$ is any choice of such embeddings from different pairs, and

$$D(x) \;=\; \Im\bigl(Li_2(x) + \log(1-x)\log|x|\bigr)$$

is the Bloch-Wigner dilogarithm function. It vanishes on all combinations in (4.2).

Let $(Q_1, \dots, Q_r)$ be an arbitrary solution of the system of algebraic equations (1.2) in some number field K. Then the element $[Q_1] + \dots + [Q_r] \in \mathbb{Z}[K^* \setminus 1]$ belongs to the kernel of (4.1). Indeed, we have

$$\sum_i Q_i \wedge (1-Q_i) \;=\; \sum_i Q_i \wedge \prod_j Q_j^{A_{ij}} \;=\; \sum_{i,j} A_{ij} Q_i \wedge Q_j \;=\; 0$$

because of the symmetry $A_{ij} = A_{ji}$. Hence every solution of (1.2) defines an element in the Bloch group of the corresponding field.

Recall that there exists the unique solution $(Q_1^0, \dots, Q_r^0)$ of (1.2) with $Q_i^0 \in (0,1)$, and we have used this solution to compute the asymptotics of (1.1) when $q \to 1$. If (1.1) is a modular function then for this solution we have

$$L(Q_1^0) + \dots + L(Q_r^0) \in \pi^2 \mathbb{Q} \tag{4.3}$$

where $L(x)$ is the Rogers dilogarithm function (condition (ii) of Corollary 3.2). Rogers dilogarithm is defined in $\mathbb{R}$ and takes values in $\pi^2\mathbb{Q}$ on all combinations of real arguments of the form (4.2). On the other hand, these are essentially all known functional equations for $L(x)$. Therefore it is very naturally to expect that $[Q_1^0] + \dots + [Q_r^0]$ is torsion in the corresponding Bloch group because of (4.3). (It is automatically torsion if the field $\mathbb{Q}(Q_1^0, \dots, Q_r^0)$ is totally real.) Similar reasoning lead Werner Nahm to the following conjecture.

Conjecture 4.1. *For a positive definite symmetric $r \times r$ matrix with rational coefficients A the following are equivalent:*

(i) *The element* $[Q_1]+\cdots+[Q_r]$ *is torsion in the corresponding Bloch group for every solution of* (1.2).
(ii) *There exist* $B \in \mathbb{Q}^r$ *and* $C \in \mathbb{Q}$ *such that* $F_{A,B,C}$ *is a modular function.*

This conjecture is true in case $r=1$, and there are a lot of examples supporting the Conjecture also for $r>1$ (see [8]). Although examples show that it is not sufficient to require only $[Q_1^0]+\cdots+[Q_r^0]$ to be torsion, it doesn't actually follow from anywhere that one should consider all solutions of (1.2) in (i). We will see soon that this requirement is indeed too strong.

As an example, let us consider matrices of the form $A=\begin{pmatrix} a & 1-a \\ 1-a & a\end{pmatrix}$. The corresponding equations are

$$\begin{cases} 1-Q_1 \;=\; Q_1^a\, Q_2^{1-a}\,, \\ 1-Q_2 \;=\; Q_1^{1-a} Q_2^a\,, \end{cases}$$

hence

$$\begin{aligned} \frac{1-Q_1}{Q_2} \;&=\; \Big(\frac{Q_1}{Q_2}\Big)^a \;=\; \frac{Q_1}{1-Q_2}\,, \\ (1-Q_1)(1-Q_2) \;&=\; Q_1 Q_2\,, \\ Q_1+Q_2 \;=\; 1 \quad &\Rightarrow \quad [Q_1]+[Q_2] \;=\; 0 \text{ in } B(\mathbb{C})\,. \end{aligned}$$

This computation is the same for all values of a and we see from Table 2 that indeed we have modular functions for every a.

Next, let us look at the table from Theorem 3.4. One can check that the matrix $A=\begin{pmatrix} 1 & -1/2 \\ -1/2 & 1\end{pmatrix}$ satisfies condition (i) of the Conjecture. (All solutions of (1.2) are $(Q_1,Q_2)=(x,x)$ with $1-x=x^{1/2}$.) However, $A=\begin{pmatrix} 3/4 & -1/4 \\ -1/4 & 3/4\end{pmatrix}$ does not satisfy (i), and so we get a counterexample to Nahm's conjecture, since there do exist corresponding modular functions. Indeed, consider the corresponding equation:

$$\begin{cases} 1-Q_1 \;=\; Q_1^{3/4}\, Q_2^{-1/4}\,, \\ 1-Q_2 \;=\; Q_1^{-1/4} Q_2^{3/4}\,. \end{cases} \tag{4.4}$$

It is algebraic equation in the variables $Q_1^{1/4}$ and $Q_2^{1/4}$. Let $t=Q_1^{1/4}Q_2^{-1/4}$. Then we have from the above equations

$$\begin{aligned} \frac{1-Q_1}{Q_2^{1/2}} \;&=\; t^3 \qquad \Rightarrow \quad Q_2^{1/2} \;=\; t^{-3}(1-Q_1)\,, \\ \frac{1-Q_2}{Q_1^{1/2}} \;&=\; t^{-3} \qquad \Rightarrow \quad Q_1^{1/2} \;=\; t^{3}(1-Q_2)\,, \end{aligned}$$

and we substitute these equalities into $Q_1^{1/2}=t^2Q_2^{1/2}$ to get

$$\begin{aligned} t^3(1-Q_2) \;&=\; t^2 t^{-3}(1-Q_1)\,, \\ t^4(1-Q_2) \;&=\; 1\,-\,Q_1 \;=\; 1\,-\,t^4 Q_2\,, \\ t^4 \;&=\; 1\,. \end{aligned}$$

Consequently, all solutions of (4.4) are $(Q_1, Q_2) = (x, x)$ where x is a solution of $1 - x = tx^{1/2}$ for a 4th root of unity $t^4 = 1$. Equivalently,

$$(1-x)^4 = x^2 \quad \Leftrightarrow \quad (x^2 - 3x + 1)(x^2 - x + 1) = 0\,.$$

We see that $(Q_1, Q_2) = \left(\frac{1+\sqrt{-3}}{2}, \frac{1+\sqrt{-3}}{2}\right)$ is a solution of (4.4), and the corresponding element $2\left[\frac{1+\sqrt{-3}}{2}\right]$ is not torsion because $D\left(\frac{1+\sqrt{-3}}{2}\right) = 1.01494....$ Here D is the Bloch-Wigner dilog (see [8, Chapter I, Section 3]) for which it is known that $D(x) = 0$ if and only if $x \in \mathbb{R}$.

A similar thing happens in Table 1: the matrix $A = \begin{pmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{pmatrix}$ satisfies the Conjecture while $A = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix}$ is a counterexample. So far we have two counterexamples, and we notice that both matrices match into the following general pattern.

Theorem 4.2. *Let A be a real positive definite symmetric $r \times r$-matrix, B a vector of length r, and C a scalar. For an arbitrary $m \geq 1$ we define*

$$A' = \mathrm{I}_{mr} + \mathrm{E}_m \otimes (A - \mathrm{I}_r)\,, \quad B' = l_{mr} + e_m \otimes (B - l_r)\,, \quad C' = C/m,$$

where $E_m \in M_{m\times m}(\mathbb{Q})$ such that $(E_m)_{ij} = 1/m$, $e_m \in \mathbb{Q}^m$ such that $(e_m)_i = 1/m$ and $l_r \in \mathbb{Q}^r$ such that $(l_r)_i = \frac{2i-r-1}{2r}$. Then

$$F_{A',B',C'}(q) = F_{A,B,C}(q^{1/m})\,.$$

Proof. The proof relies on the following identity

$$\frac{q^{\frac{1}{2}n^2}}{(q)_n} = \sum_{\substack{k\in(\mathbb{Z}_{\geq 0})^m \\ k_1+\ldots+k_m=n}} \frac{q^{\frac{m}{2}k^Tk+ml_m^Tk}}{(q^m;q^m)_{k_1}\cdots(q^m;q^m)_{k_m}}$$

which holds for all $n \geq 0$. It follows directly if we use (3.4) on both sides in the trivial identity

$$(-xq^{1/2};q)_\infty = (-xq^{1/2};q^m)_\infty(-xq^{3/2};q^m)_\infty\cdots(-xq^{m-1/2};q^m)_\infty,$$

and compare the coefficient of x^n on both sides.

Using the identity we find

$$\begin{aligned} F_{A,B,C}(q) &= \sum_{n\in(\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}n^TAn+n^TB+C}}{(q)_{n_1}\ldots(q)_{n_r}} \\ &= \sum_{n\in(\mathbb{Z}_{\geq 0})^r} q^{\frac{1}{2}n^T(A-\mathrm{I}_r)n+n^TB+C} \sum_{\substack{K\in M_{r\times m}(\mathbb{Z}_{\geq 0}) \\ mKe_m=n}} \frac{q^{\frac{m}{2}||K||^2+mre_r^TKl_m}}{(q^m;q^m)_K}, \end{aligned}$$

where $||K||^2 = \sum_{i=1}^r\sum_{j=1}^m K_{ij}^2$ and $(q;q)_K = \prod_{i=1}^r\prod_{j=1}^m(q;q)_{K_{ij}}$. Now changing the order of summation we get that this equals

$$\sum_{K\in M_{r\times m}(\mathbb{Z}_{\geq 0})} \frac{q^{\frac{m^2}{2}e_m^TK^T(A-\mathrm{I}_r)Ke_m+\frac{m}{2}||K||^2+me_m^TK^TB+mre_r^TKl_m+C}}{(q^m;q^m)_K}.$$

If we turn the $r \times m$ matrix K into a vector of length rm by putting the columns of K under each other, we can recognize this last sum as

$F_{A',B'',C'}(q^m)$, where A' and C' are as in the theorem and $B'' = e_m \otimes B + rl_m \otimes e_r$. We can easily verify that

$$rl_m \otimes e_r = l_{mr} - e_m \otimes l_r,$$

which gives $B'' = B'$, with B' as in the theorem. So we have found

$$F_{A,B,C}(q) = F_{A',B',C'}(q^m).$$

Now replacing q by $q^{1/m}$ gives the desired result. □

Let us take $r = 1$ and $m = 2$. Then

$$A = \frac{1}{2} \quad \rightsquigarrow \quad A' = \begin{pmatrix} 3/4 & -1/4 \\ -1/4 & 3/4 \end{pmatrix}$$
$$A = 2 \quad \rightsquigarrow \quad A' = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix}$$

and the theorem produces modular functions for these 2×2 matrices from the ones known for $r = 1$. One can construct more counterexamples with higher r using Theorem 4.2.

Finally, we would like to give one more counterexample, this time such that A has integer entries. Let

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad C = \frac{1}{15}.$$

All solutions of (1.2) in this case are

$$(Q_1, Q_2, Q_3, Q_4) = \left(u, u, \frac{1}{1+u}, \frac{1}{1+u}\right) \text{ with } 1 - u^2 = u^4$$

and

$$(Q_1, Q_2, Q_3, Q_4) = \left(u, -u, \frac{1}{1+u}, \frac{1}{1-u}\right) \text{ with } 1 - u^2 = -u^4.$$

It is easy to check that solutions of the first type give torsion elements in the Bloch group, while ones of the second type give non-torsion elements. On the other hand, we have that

$$F_{A,B,C}(q) = \frac{\eta(2z)^2 \theta_{5,1}(z)}{\eta(z)^3}.$$

We get this identity by applying the theorem below to $A = \left(\begin{smallmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 1/4 \\ -1/4 \end{smallmatrix}\right)$ and $C = -1/120$, and using the identity for this case given in Table 1.

Theorem 4.3. *Let A be a real positive definite symmetric $r \times r$-matrix, B a vector of length r, and C a scalar. Let A', B' and C' be the symmetric $2r \times 2r$-matrix, the vector of length $2r$ and the scalar, resp. , given by*

$$A' = \begin{pmatrix} 2A & \mathrm{I}_r \\ \mathrm{I}_r & \mathrm{I}_r \end{pmatrix}, \quad B' = \begin{pmatrix} 2B \\ \frac{1}{2} \\ \vdots \\ \frac{1}{2} \end{pmatrix}, \quad C' = 2C + \frac{r}{24},$$

then

$$F_{A',B',C'}(q) = \frac{\eta(2z)^r}{\eta(z)^r} F_{A,B,C}(q^2).$$

Proof. Using $(q^2;q^2)_n = (q;q)_n(-q;q)_n$, $(q^2;q^2)_\infty = (q;q)_\infty(-q;q)_\infty$ and (3.4), we see that

$$\frac{(q^2;q^2)_\infty}{(q;q)_\infty}\frac{1}{(q^2;q^2)_n} = \frac{(-q;q)_\infty}{(q;q)_n(-q;q)_n} = \frac{(-q^{n+1};q)_\infty}{(q;q)_n} = \frac{1}{(q)_n}\sum_{k\geq 0}\frac{q^{\frac{1}{2}k^2+\frac{1}{2}k+nk}}{(q)_k},$$

and so

$$\begin{aligned}\frac{(q^2;q^2)_\infty^r}{(q)_\infty^r}&F_{A,B,C}(q^2)\\ &= \sum_{n\in(\mathbb{Z}\geq 0)^r}\frac{q^{n^TAn+2n^TB+2C}}{(q)_{n_1}\cdots(q)_{n_r}}\sum_{k\in(\mathbb{Z}\geq 0)^r}\frac{q^{\frac{1}{2}k^Tk+n^Tk+\frac{1}{2}(k_1+k_2+\ldots+k_r)}}{(q)_{k_1}\cdots(q)_{k_r}}\\ &= \sum_{n,k\in(\mathbb{Z}\geq 0)^r}\frac{q^{n^TAn+\frac{1}{2}k^Tk+n^Tk+2n^TB+\frac{1}{2}(k_1+k_2+\ldots+k_r)+2C}}{(q)_{n_1}\cdots(q)_{n_r}(q)_{k_1}\cdots(q)_{k_r}}.\end{aligned}$$

If we turn the two vectors n and k into one vector of length $2r$ by putting k below n, we can recognize this last sum as $q^{-r/24}F_{A',B',C'}(q)$, where A', B' and C' are as in the theorem. So we have found

$$\frac{(q^2;q^2)_\infty^r}{(q)_\infty^r}F_{A,B,C}(q^2) = q^{-r/24}F_{A',B',C'}(q).$$

Multiplying both sides by $q^{r/24}$ gives the desired result. □

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